Eigenvalues of Hadamard powers of large symmetric Pascal matrices

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Received 22 June 2004; accepted 10 February 2005
Available online 5 May 2005
Submitted by B.L. Shader

Abstract

Let $S_n$ be the positive real symmetric matrix of order $n$ with $(i, j)$ entry equal to $\binom{i + j - 2}{j - 1}$, and let $x$ be a positive real number. Eigenvalues of the Hadamard (or entry wise) power $S_n^{(x)}$ are considered. In particular for $k$ a positive integer, it is shown that both the Perron root and the trace of $S_n^{(k)}$ are approximately equal to $\frac{x^k}{x^2-1} \left( \frac{2n-2}{n-1} \right)^k$.

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Keywords: Eigenvalues; Symmetric Pascal matrices; Real Hadamard powers; Perron roots; Positive matrices; Positive definite matrices; Convergence; MATLAB

1. Introduction

The symmetric Pascal matrix of order $n$ is the real symmetric matrix $S_n = (s_{ij})$ with

$$s_{ij} = \binom{i + j - 2}{j - 1}.$$
\[ s_{ij} = \begin{pmatrix} i + j - 2 \\ j - 1 \end{pmatrix} \quad \text{for } i, j = 1, 2, \ldots, n. \]

Since \( S_n \) can be factored as \( S_n = U_n^T U_n \) where \( U_n \) is an involutory matrix [1], it is easy to see that the eigenvalues of \( S_n \) have a number of special properties. For example, \( S_n \) has \( n \) distinct positive eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \), and \( \{1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n\} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \).

We wondered what could be said about the maximal eigenvalue (that is, the Perron root) of the positive matrix \( S_n \). Using MATLAB, it was seen that the Perron root of \( S_n \) increased quite rapidly as \( n \) increased. With such a rapid increase, it did not seem possible to find a recognizable pattern for these values. In order to get more revealing numbers, it was decided to transform \( S_n \) by multiplying it by a rapidly decreasing positive function of \( n \). The chosen function for this transformation was the reciprocal of the \((n, n)\) entry of \( S_n \). Thus we considered the Perron root \( \mu_n \) of the regularized symmetric Pascal matrix \( R_n = \begin{pmatrix} 2n - 2 \\ n - 1 \end{pmatrix}^{-1} S_n \). Although the use of this particular regularization was based more on convenience than insight, it was found to be an excellent choice. Both \( \mu_n \) and the trace \( \tau_n \) of \( R_n \) seemed to be converging. Based on data such as that found in Table 1, we conjectured that \( \lim_{n \to \infty} \mu_n = \lim_{n \to \infty} \tau_n = \frac{4}{3} \). It was not difficult to prove the conjecture for \( \tau_n \), but a proof of the one for \( \mu_n \) was more elusive. Fortunately MATLAB computations lead to a diagonal matrix \( D \) that yielded a useful lower bound for the row sums of the matrix \( D^{-1} R_n D \). Thus \( \lim_{n \to \infty} \mu_n = \lim_{n \to \infty} \tau_n = \frac{4}{3} \), which implies that for large values of \( n \) the positive definite matrix \( S_n \) has a dominant eigenvalue that is much larger than the sum of all of the \( n - 1 \) other eigenvalues.

These results have natural extensions to Hadamard powers. Let \( x \) be a real number and let \( A = (a_{ij}) \) be a nonnegative matrix of order \( n \). The matrix \( A^{(x)} = (a_{ij}^x) \) of order \( n \) obtained by raising each entry of \( A \) to the power \( x \) is a Hadamard power of \( A \). For each \( x > 0 \), let \( \mu_n(x) \) and \( \tau_n(x) \), respectively, denote the Perron root and the trace of the Hadamard power \( R_n^{(x)} \).

In Section 2, it is shown that \( \lim_{n \to \infty} \inf \mu_n(x) \geq \frac{4^x}{3^{x-1}} \) for each \( x > 0 \). In the next section, it is shown that \( \lim_{n \to \infty} \tau_n(x) = \frac{4^x}{3^{x-1}} \) for each \( x > 0 \), and that

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<th>( \tau_n )</th>
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<tr>
<td>140</td>
<td>1.3328</td>
<td>1.3349</td>
</tr>
</tbody>
</table>

Table 1

Perron root and trace of \( R_n \)
\[ \lim_{n \to \infty} \mu_n(k) = \frac{4^k}{4^k - 1} \] when \( k \) is a positive integer. The paper concludes with two equivalent conjectures and a brief look at a related matrix.

2. A lower bound for the Perron root

The following lemmas will be used in obtaining a bound on the lower limit of the sequence \( \{\mu_n(x)\} \) when \( x > 0 \). For \( i \) and \( n \) positive integers with \( i \leq n \), let

\[ r_{ni}(x) = \frac{1}{\left(\frac{n+i-2}{n-1}\right)^x \left(\frac{2n-2}{n-1}\right)^x} \sum_{j=1}^{n} \left(\frac{i+j-2}{j-1}\right)^x \left(\frac{n+j-2}{j-1}\right)^x. \]

**Lemma 2.1.** For each \( x > 0 \) and integer \( n \geq 2 \), \( r_{n1}(x) > r_{n2}(x) > \cdots > r_{nn}(x) \).

**Proof.** Let \( x > 0 \) and \( n \geq 2 \). Let

\[ w_{ij}(x) = \left(\frac{i+j-2}{j-1}\right)^x \left(\frac{n+i-2}{n-1}\right)^x \] for \( 1 \leq i, j \leq n \).

Using the convention that an empty product is equal to 1, we see that

\[ w_{ij}(x) = \prod_{k=2}^{i} \left(\frac{i+j-k}{i+n-k}\right)^x \] for \( 1 \leq i, j \leq n \).

Thus \( w_{in}(x) = 1 \) for \( 1 \leq i \leq n \), and

\[ w_{1j}(x) > w_{2j}(x) > \cdots > w_{nj}(x) \] for \( 1 \leq j < n \).

Therefore, since

\[ r_{ni}(x) = \frac{1}{\left(\frac{2n-2}{n-1}\right)^x} \sum_{j=1}^{n} w_{ij}(x) \left(\frac{n+j-2}{j-1}\right)^x \] for \( 1 \leq i \leq n \),

it follows that \( r_{n1}(x) > r_{n2}(x) > \cdots > r_{nn}(x) \). \( \square \)

We now write \( r_n(x) = r_{nn}(x) \) and consider the sequence \( \{r_n(x)\} \).

**Lemma 2.2.** For each \( x > 0 \), there exists a positive number \( L_x \) such that

\[ \lim_{n \to \infty} r_n(x) = L_x \leq \frac{4^x}{4^x - 1}. \]
Proof. Let \( x > 0 \) and let \( n \) be a positive integer. We have

\[
r_n(x) = \frac{1}{(2n^2 - 2)} \sum_{j=1}^{n} \left( \frac{n + j - 2}{j - 1} \right)^{2x}
\]

\[
= \sum_{k=1}^{n} \left( \frac{2n - k - 1}{n - k} \right)^{2x} \left( \frac{2n - 2}{n - 1} \right)^{2x}
\]

\[
= \sum_{k=1}^{n} k \left( \frac{n + 1 - i}{2n - i} \right)^{2x}.
\]

Hence, for \( n = 1, 2, \ldots \), it follows that \( r_n(x) < r_{n+1}(x) \) and

\[
r_n(x) \leq \sum_{k=1}^{n} \left( \frac{1}{4^x} \right)^{k-1} = \frac{4^x \left( 1 - \left( \frac{1}{4^x} \right)^n \right)}{4^x - 1}.
\]

Thus \( \{r_n(x)\} \) is a bounded increasing sequence, and

\[
\lim_{n \to \infty} r_n(x) \leq \lim_{n \to \infty} \left( \frac{4^x \left( 1 - \left( \frac{1}{4^x} \right)^n \right)}{4^x - 1} \right) = \frac{4^x}{4^x - 1}.
\]

□

Lemma 2.3. For each \( x > 0 \),

\[
\lim_{n \to \infty} r_n(x) = \frac{4^x}{4^x - 1}.
\]

Proof. Let \( x > 0 \). The subsequence \( \{r_{n^2}(x)\} \) of \( \{r_n(x)\} \) also converges to \( L_x \) of Lemma 2.2. However, we see that for each positive integer \( n \),

\[
r_{n^2}(x) \geq \sum_{k=1}^{n} \left( \frac{2n^2 - k - 1}{n^2 - k} \right)^{2x} \left( \frac{2n^2 - 2}{n^2 - 1} \right)^{2x}
\]

\[
\geq \sum_{k=1}^{n} \left( \frac{n^2 - n + 1}{2n^2 - n} \right)^{2x} \left( \frac{2n^2 - n}{n^2 - n + 1} \right)^{2x} \left( 1 - \left( \frac{n^2 - n + 1}{2n^2 - n} \right)^{2x} \right)^n
\]

\[
= \frac{\left( \frac{2n^2 - n}{n^2 - n + 1} \right)^{2x} \left( 1 - \left( \frac{n^2 - n + 1}{2n^2 - n} \right)^{2x} \right)^n}{\left( \frac{2n^2 - n}{n^2 - n + 1} \right)^{2x} - 1},
\]

\[
\lim_{n \to \infty} r_{n^2}(x) = \frac{4^x}{4^x - 1}.
\]
and it follows that
\[
\frac{4^x}{4^x - 1} \geq \lim_{n \to \infty} r_n(x) = L_x = \lim_{n \to \infty} r_n^2(x) \geq \frac{4^x}{4^x - 1}. \quad \Box
\]

**Theorem 2.4.** For each \( x > 0 \),
\[
\lim_{n \to \infty} \inf \mu_n(x) \geq \frac{4^x}{4^x - 1}.
\]

**Proof.** Let \( x > 0 \), let \( D \) be the diagonal matrix of order \( n \) whose diagonal entries are the entries of the last column of \( S_n^{(x)} \), and let \( B = D^{-1} R_n(x) D \). Then the positive matrix \( B \) has Perron root \( \mu_n(x) \) and row sums \( r_{ni}(x) \) for \( i = 1, 2, \ldots, n \). The theorem now follows from Lemmas 2.1 and 2.3, since it is well known (for example, see [4]) that the Perron root \( \mu_n(x) \) is at least as large as the minimal row sum \( r_n(x) \). \( \Box \)

3. The trace and positive integer Hadamard powers

We now consider the sequence \( \{ \tau_n(x) \} \) when \( x > 0 \).

**Lemma 3.1.** For each \( x > 0 \),
\[
\tau_{n+1}(x) = 1 + \left( \frac{1}{2(2 - 1/n)} \right)^x \tau_n(x) \quad \text{for } n = 1, 2, \ldots
\]

**Proof.** Let \( x > 0 \). For each positive integer \( n \), we have
\[
\begin{align*}
\tau_{n+1}(x) &= \frac{1}{\binom{2n}{n}} \text{trace}(S_{n+1}^{(x)}) \\
&= 1 + \frac{1}{\binom{2n}{n}} \text{trace}(S_n^{(x)}) \\
&= 1 + \frac{1}{\binom{2n}{n}} \frac{(2n - 2)^x}{(n - 1)} \tau_n(x) \\
&= 1 + \left( \frac{1}{\frac{2n}{2(2 - 1/n)}} \right)^x \tau_n(x). \quad \Box
\end{align*}
\]

**Lemma 3.2.** For each \( x > 0 \), one of the following holds:

(a) \( \tau_n(x) < \tau_{n+1}(x) \) for \( n = 1, 2, \ldots \).
(b) there exists a positive integer \( m \) such that \( \tau_n(x) > \tau_{n+1}(x) \) for \( n \geq m + 1 \).
Proof. Let \( x > 0 \). Suppose that (a) does not hold. Then there exists a positive integer \( m \) such that \( \tau_m(x) \geq \tau_{m+1}(x) \). Using induction and Lemma 3.1, we see that \( \tau_n(x) > \tau_{n+1}(x) \) for all integers \( n \geq m + 1 \). □

Theorem 3.3. For each \( x > 0 \),
\[
\lim_{n \to \infty} \tau_n(x) = \frac{4^x}{4^x - 1}.
\]
Proof. Let \( x > 0 \). Clearly \( \{\tau_n(x)\} \) is a bounded sequence, and Lemma 3.2 implies that this sequence is monotone for sufficiently large \( n \). Thus \( \lim_{n \to \infty} \tau_n(x) \) exists, and the theorem now follows from Lemma 3.1. □

We now consider positive integer Hadamard powers of \( R_n \).

Theorem 3.4. For each positive integer \( k \),
\[
\lim_{n \to \infty} \mu_n(k) = \frac{4^k}{4^k - 1}.
\]
Proof. Let \( k \) be a positive integer. Since positive integer Hadamard powers of symmetric positive definite matrices are positive definite (for example, see [3]), \( R_n^{(k)} \) is positive definite. Thus \( \mu_n(k) \leq \tau_n(k) \) for \( n = 1, 2, \ldots \), and the theorem follows from Theorems 2.4 and 3.3. □

Let \( k \) be a positive integer. Theorems 3.3 and 3.4 imply that for large values of \( n \) the positive definite matrix \( S_n^{(k)} \) has a dominant eigenvalue that is much larger than the sum of all of its \( n - 1 \) other eigenvalues.

4. Conjectures and a related matrix

Can Theorem 3.4 be extended to other Hadamard powers of \( R_n \)? We propose the following.

Conjecture 4.1. For each positive real number \( x \), \( \lim_{n \to \infty} \mu_n(x) = \frac{4^x}{4^x - 1} \).

Theorem 3.3 implies that Conjecture 4.1 is equivalent to the following.

Conjecture 4.2. For each positive real number \( x \), \( \lim_{n \to \infty} \frac{\mu_n(x)}{\tau_n(x)} = 1 \).

If \( P_n = (p_{ij}) \) is the real lower triangular matrix of order \( n \) with
\[
p_{ij} = \binom{i-1}{j-1} \quad \text{for} \quad 1 \leq j \leq i \leq n,
\]
then \( S_n = P_n P_n^T \) [2]. Since the positive matrix \( \hat{S}_n = P_n^T P_n \) has the same eigenvalues as \( S_n \), Theorems 3.3 and 3.4 for \( x = k = 1 \) imply the following.

**Theorem 4.3.** If \( q_n \) and \( t_n \), respectively, denote the Perron root and trace of \( \hat{S}_n \), then

\[
\lim_{n \to \infty} q_n \left( \frac{2n - 2}{n - 1} \right) = \frac{4}{3} = \lim_{n \to \infty} t_n \left( \frac{2n - 2}{n - 1} \right).
\]

Let \( q_n(x) \) and \( t_n(x) \), respectively, denote the Perron root and trace of the Hadamard power \( \hat{S}_n^{(x)} \). Theorem 4.3 implies that \( \lim_{n \to \infty} \frac{q_n(x)}{t_n(x)} = 1 \). We propose the following.

**Problem 4.4.** Determine the positive real numbers \( x \) for which \( \lim_{n \to \infty} \frac{q_n(x)}{t_n(x)} = 1 \).

**Acknowledgment**

The authors wish to thank Dr. Reza Adhami, Professor and Chair of the Department of Electrical and Computer Engineering of the University of Alabama in Huntsville, for his support.

**References**